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The hyperspace of a compact space, I

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Abstract

We investigate the properties monolithic and d-separable for the hyperspace $H(X)$ of all nonempty closed subsets of a compact Hausdorff space X. A. Arhangelskii has asked whether $H(X)$ monolithic is equivalent to X metrizable. We answer this with: Let X be a compact orderable space. Then $H(X)$ is monolithic iff X is monolithic and hereditarily Lindelöf. So, a Suslin continuum has a monolithic hyperspace. In contrast, $\mathrm{MA}(\omega_1)$ implies that for any compact Hausdorff space X, $H(X)$ is monolithic iff X is metrizable.

We prove that $H(X)$ is *always* d-separable. A special case of this yields that every locally compact Hausdorff space X has a discrete (in $H(X)$) π-net.

Keywords: Hyperspace; Compact; Monolithic; d-separable; π-base

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1. Introduction

We are interested in the properties monolithic and d-separable for the hyperspace $H(X)$ of a compact Hausdorff space X.

Problem II.10 of Arhangelskii [4] asks "Is it true that $H(X)$ is monolithic iff X is metrizable?". We will show in Section 2 that this equivalence is independent of ZFC. Problem II.9 asks "When is $H(X)$ monolithic?". We don't have a satisfactory answer to this in ordinary set theory except for the case when X is orderable.

Two very interesting results on d-separability are Arhangelskii's result [3] that d-separable is a productive property and Amirdzhanov's result [1] that for a Tychonoff space X, $X^{\mathrm{d}(X)}$ (where $\mathrm{d}(X)$ is the density of X) is d-separable. In Section 3, we will

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show that if X is locally compact Hausdorff, then $H(X)$ is d-separable. An interesting corollary is that every locally compact Hausdorff space has a discrete (in the hyperspace) π-net consisting of sets of cardinality at most 2. Along the way we will show that $H(\omega^*)$ (where ω^* is the Stone–Čech remainder of a countably infinite discrete space) even has a σ-disjoint π-base.

The *weight* of a space X, denoted by $\mathrm{w}(X)$, is the least cardinality of a base for X. A *π-base* $\mathcal{P}$ for a space X is a collection of nonempty open subsets of X such that for every nonempty open set U, there exists $P \in \mathcal{P}$ with $P \subset U$.

Let $H(X) = \{F\colon F$ is a nonempty closed subset of $X\}$ endowed with the Vietoris topology. Our hyperspace notation is as follows: If $A \subset X$, then $[A] = \{F \in H(X)\colon F \cap A \neq \emptyset\}$ and $\langle A\rangle = \{F \in H(X)\colon F \subset A\}$. The family $\{[O], \langle O\rangle\colon O$ is open in $X\}$ is used as an open subbase for the Vietoris topology. If $\mathcal{A}$ is a collection of subsets of X, then $\langle\mathcal{A}\rangle = \{F \in H(X)\colon F \subset \bigcup\mathcal{A}$ and $F \cap A \neq \emptyset$ for every $A \in \mathcal{A}\}$. The family $\{\langle\mathcal{O}\rangle\colon \mathcal{O}$ is a finite family of open subsets of $X\}$ is an open base for $H(X)$. Basic facts are that $H(X)$ is compact iff X is compact and that $\mathrm{w}(H(X)) = \mathrm{w}(X)$. The reader is referred to Kuratowski [9] for other basic facts of $H(X)$.

In both of the following sections we will use the following facts. For a proof of (a), the reader is referred to Hodel [6]. A proof of (b) is an exercise in basic topology. Let F be a subspace of X. A collection $\mathcal{O}$ of subsets of X is said to *(strongly) T_1-separate* the points of F if for each pair of distinct points x, y of F, there exists $O \in \mathcal{O}$ with $x \in O$ and $(y \notin \overline{O})$ $y \notin O$.

Fact 1.1. *Let F be a closed subset of a compact Hausdorff space X and let $\mathcal{B}$ be an open base of X. Then,*

(a) *if there exists a collection $\mathcal{O}$ of open subsets of X such that $\mathcal{O}$ T_1-separates the points of F, then $\mathrm{w}(F) \leqslant |\mathcal{O}|$,*

(b) *there exists a subcollection $\mathcal{O} \subset \mathcal{B}$ with $|\mathcal{O}| \leqslant \mathrm{w}(F)$ such that $\mathcal{O}$ strongly T_1-separates the points of F.*

2. Monolithicity of the hyperspace

A *net* $\mathcal{N}$ for a space X is a collection of subsets of X such that for every $x \in X$ and for every open set U with $x \in U$, there exists $N \in \mathcal{N}$ with $x \in N \subset U$. The *net weight* of a space X, denoted by $\mathrm{nw}(X)$, is the least cardinality of a net for X.

Definition 2.1 (Arhangelskii [2]). X is *monolithic* if $\mathrm{nw}(\overline{A}) \leqslant |A|$ for all $A \subset X$. Monolithic is an hereditary and ω-productive property. For compact Hausdorff spaces X, monolithic means that $\mathrm{w}(\overline{A}) \leqslant |A|$ for all $A \subset X$.

The following proposition and corollary follow from well-known results and so we assume that they are known, but we include proofs for the reader's benefit. We use the abbreviation HL for Hereditarily Lindelöf. Note that for compact Hausdorff spaces, HL is equivalent to perfectly normal.

Proposition 2.2. *Let X be a T_1 space. If $H(X)$ is monolithic then X is monolithic, HL and compact.*

Proof. Assume that $H(X)$ is monolithic. X is embeddable in $H(X)$ as the singletons and monolithic is hereditary so X is monolithic. If X were not countably compact, then let N be a countably infinite discrete closed subspace of X. Thus $H(N)$ is embeddable in $H(X)$, but $H(N)$ is not monolithic (it is separable yet has an uncountable discrete subset so it cannot have a countable net). Hence, X is countably compact. Let us assume that X is not HL. We can choose $S = \{x_\alpha\colon \alpha < \omega_1\} \subset X$ such that for each $\beta < \omega_1$, $\{x_\alpha\colon \alpha < \beta\}$ is open in S. Since S cannot have a countable net, S is not separable. By induction, we can construct an uncountable $A \subset \omega_1$ such that $D = \{x_\alpha\colon \alpha \in A\}$ is a discrete subspace. Now, put $D' = \overline{D} \setminus D$ and consider the subspace $K = \{D' \cup E\colon E \subset D\}$ of $H(X)$. The map $\varphi : 2^D \to K$ defined by $\varphi(f) = D' \cup f^{-1}(1)$ is a continuous surjection (to prove continuity note that since X is countably compact, any open set $U \subset X$ such that $D' \subset U$ satisfies that $D \setminus U$ is finite). Hence, K is separable and therefore has a countable net. But this is impossible since $\{D' \cup \{x_\alpha\}\colon \alpha \in A\}$ is an uncountable discrete subspace of K. We conclude that X is HL. So, X is compact as well. □

Corollary 2.3 (MA(ω_1)). *Let X be a compact Hausdorff space. $H(X)$ is monolithic iff X is metrizable.*

Proof. For the nontrivial direction, assume that $H(X)$ is monolithic. By Proposition 2.2 X is a monolithic and HL compact space. MA(ω_1) implies that all first countable, ccc, compact Hausdorff spaces are separable (Juhasz [7]). Hence X has countable weight and therefore is metrizable. □

Question 2.4. For compact Hausdorff X, is $H(X)$ monolithic iff X is monolithic and HL?

Currently the main examples of nonmetrizable, monolithic and HL compact Hausdorff spaces are a Suslin continuum and Kunen's compact L-space [8]. The former we shall see (Corollary 2.8) has a monolithic hyperspace while we have been unable to resolve the status of the latter. The question remains unresolved.

For $(X, \prec)$ a compact ordered space, we will use the following notations for the remainder of this section. The closed interval $[a, b] = \{x \in X\colon a \preceq x \preceq b\}$ and the other standard interval notations $(a, b]$, $[a, b)$ and (a, b). If $x \in X$ and $H \subset X$ and for every $y \in H$ we have that $x \prec y$, then we write $x \prec H$ (similarly we define $x \succ H$). A *jump* of X is a $\{x \prec y\}$ such that $(x, y) = \emptyset$. Every closed subset F of X has a maximum and a minimum element denoted by $\max(F)$ and $\min(F)$. We let $1 = \max(X)$ and $0 = \min(X)$. We put $\mathcal{B}(X) = \{F\colon F$ is a union of finitely many closed intervals of $X\}$. For each $F \in \mathcal{B}(X)$ we define the set $e(F)$ of endpoints of F as follows: If $F = [a, b]$, then $e(F) = \{a, b\}$. In general, we put $e(F) = \bigcup\{e(G)\colon G$ is a maximal closed interval of X contained in $F\}$. Then, for each $F \in \mathcal{B}(X)$, $e(F)$ is a finite set.

Lemma 2.5. *Let $(X, \prec)$ be a compact ordered space and let $\mathcal{C} \subset \mathcal{B}(X)$. Put $E = \bigcup_{K \in \mathcal{C}} e(K)$. If $F \in \overline{\mathcal{C}}^{H(X)}$, then*

(a) $\{\min(F), \max(F)\} \subset \overline{E}$,

(b) *if $x \notin F$ and $y \in F \cap [x, 1]$, then* $\min(F \cap [x, 1]) \in \overline{E}$,

(c) *if $x \notin F$ and $y \in F \cap [0, x]$, then* $\max(F \cap [0, x]) \in \overline{E}$.

Proof. We will only prove (b) as (a) and (c) are similar. Let I be an open interval of X containing $f = \min(F \cap [x, 1])$ such that $I \subset (x, 1]$. $F \in [I] \cap \langle I \cup (f, 1] \cup [0, x) \rangle$ implies that we can choose $J \in \mathcal{C}$ such that $J \cap I \neq \emptyset$ and $J \subset I \cup (f, 1] \cup [0, x)$. Let K be a maximal closed interval contained in J such that $K \cap I \neq \emptyset$. Then $\min(K) \in I \cap E$. Thus, $f \in \overline{E}$. $\square$

Theorem 2.6. *Let $(X, \prec)$ be a monolithic compact ordered space. Then, if $\mathcal{C} \subset \mathcal{B}(X)$, then* $\mathrm{w}(\overline{\mathcal{C}}^{H(X)}) \leqslant |\mathcal{C}|$.

Proof. Let $\kappa = |\mathcal{C}|$ and put $E = \bigcup_{K \in \mathcal{C}} e(K)$. Since X is monolithic, $\mathrm{w}(F) \leqslant \kappa$ where $F = \overline{E}^X$. Invoke Fact 1.1(b) and let $\mathcal{A}$ be a family of open intervals of X such that $|\mathcal{A}| \leqslant \kappa$ and $\mathcal{A}$ strongly T_1-separates the points of F. We also can assume that the family $\mathcal{A}$ is closed under finite intersections. Since F is closed, the restricted order topology on F agrees with the subspace topology that F inherits from X. F, with the restricted order, can have at most κ many jumps because $\mathrm{w}(F) \leqslant \kappa$. For each jump $J = \{x \prec y\}$ of F such that J is *not* a jump of X, choose a nonempty open interval $I(J)$ of X such that $\overline{I(J)}^X \subset (x, y)$. Let $\mathcal{D} = \{I(J)\colon J \text{ is a jump of } F\}$. We finish the proof (using Fact 1.1(a)) by showing that the open family $\mathcal{O} = \{[I], \langle X \backslash \overline{I} \rangle\colon I \in \mathcal{A} \cup \mathcal{D}\}$ T_1- separates the points of $\overline{\mathcal{C}}^{H(X)}$.

To this end, let $G, H \in H(X)$ with $G \neq H$ and $\{G, H\} \subset \overline{\mathcal{C}}^{H(X)}$. W.l.o.g. assume that $G \backslash H \neq \emptyset$.

Case 1: *$\exists g \in G \backslash H$ such that $g \prec H$.* By Lemma 2.5, $g = \min(G) \in F$ and $h = \min(H) \in F$ and $g \prec h$. Let $I \in \mathcal{A}$ be such that $g \in I$ and $h \notin \overline{I}$. Then $G \in [I] \not\ni H$ and $H \in \langle X \backslash \overline{I} \rangle \not\ni G$.

Case 2: *$\exists g \in G \backslash H$ such that $g \succ H$.* Same as Case 1 with min replaced by max and $\prec$ replaced by $\succ$.

Case 3: *Not Case* 1 *and not Case* 2. Choose $g \in G \backslash H$ such that $h_0 = \max(H \cap [0, g]) \prec g \prec \min(H \cap [g, 1]) = h_1$. Lemma 2.5 implies that $\{h_0, h_1\} \subset F$. If $g \in F$, then get $I_0 \in \mathcal{A}$ such that $g \in I_0$ and $h_0 \notin \overline{I_0}$ and get $I_1 \in \mathcal{A}$ such that $g \in I_1$ and $h_1 \notin \overline{I_1}$. Put $I = I_0 \cap I_1$. Then $I \in \mathcal{A}$ and $g \in I$ and $\{h_0, h_1\} \cap \overline{I} = \emptyset$. Hence, $G \in [I] \not\ni H$ and $H \in \langle X \backslash \overline{I} \rangle \not\ni G$. Thus we assume that $g \notin F$. Thus $g \neq \min(G)$ and $g \neq \max(G)$. Put $f_0 = \max(F \cap [0, g))$ and $f_1 = \min(F \cap (g, 1])$. We have that $h_0 \preceq f_0 \prec g \prec f_1 \preceq h_1$. $\{f_0 \prec f_1\}$ is a jump of F that is *not* a jump of X (because of g). Choose $I \in \mathcal{D}$ such that $\overline{I} \subset (f_0, f_1)$. If $G \cap I = \emptyset$, then Lemma 2.5 would imply that $F \cap (f_0, f_1) \neq \emptyset$ which is not so. Hence, $G \in [I] \not\ni H$ and $H \in \langle X \setminus \overline{I} \rangle \not\ni G$. $\square$

Corollary 2.7. *Let X be a compact orderable space. Then, $H(X)$ is monolithic iff X is monolithic and HL.*

Proof. Let X be monolithic and HL and let $\mathcal{D} \subset H(X)$. Since every closed subset of X is a G_δ-set, for each $D \in \mathcal{D}$ choose a countable chain $D^* \subset \mathcal{B}(X)$ with $D = \bigcap D^*$. Put $\mathcal{C} = \bigcup_{D \in \mathcal{D}} D^*$. Then $\overline{\mathcal{D}}^{H(X)} \subset \overline{\mathcal{C}}^{H(X)}$ and $|\mathcal{D}| = |\mathcal{C}|$. Theorem 2.6 implies that $\mathrm{w}(\overline{\mathcal{C}}^{H(X)}) \leqslant |\mathcal{C}|$. Hence, $\mathrm{w}(\overline{\mathcal{D}}^{H(X)}) \leqslant |\mathcal{D}|$ and so $H(X)$ is monolithic. $\square$

Corollary 2.8 ($\neg$SH). *There exists a nonmetrizable compact orderable space with a monolithic hyperspace.*

Proof. We are assuming the negation of Suslin's Hypothesis, i.e., that there exists a ccc nonseparable orderable space. Let X be a Suslin continuum. Then X is a compact HL orderable space. Connectedness of X implies that X is monolithic as follows: Let $A \subset X$. Then $\overline{A}$ has at most countably many jumps, since each jump contains points of X and X is ccc. That $\mathrm{w}(\overline{A}) \leqslant |A|$ follows from the fact that the weight of an ordered space equals its density plus the number of jumps it has. Thus, by the preceding corollary, X has a monolithic hyperspace. $\square$

3. d-separability of the hyperspace

Definition 3.1 (Kurepa [10]). X is *d-separable* if X has a dense σ-discrete subspace. That is, X has a dense subspace which is the union of countably many discrete subspaces.

A *σ-disjoint π-base* is a π-base which is the union of countably many disjoint subcollections. A *tree π-base* T for a space X is a π-base for X such that:

(1) every 2 elements of T are either disjoint or comparable by inclusion,

(2) for every $t \in T$, $\{s \in T\colon\ s \supsetneq t\}$ is well-ordered by $\supsetneq$.

Let T be a tree π-base for X. For each $t \in T$, put $o(t) =$ the ordinal α such that α is isomorphic to $\{s \in T\colon\ s \supsetneq t\}$. For each ordinal α, put $T_\alpha = \{t \in T\colon\ o(t) = \alpha\}$. Define the height of T by $\mathrm{ht}(T) = \min\{\alpha\colon\ T_\alpha = \emptyset\}$.

It is easy to see that if X has a σ-disjoint π-base, then X has a tree π-base of height $\leqslant \omega$ and X is d-separable. A good reference for results on σ-disjoint π-bases and tree π-bases is Williams [11].

Let X be a T_1 space. Let $F(X) = \{F\colon\ F$ is a nonempty finite subset of $X\}$. Then $F(X)$ is a dense subspace of $H(X)$. For both properties P of $H(X)$ that we are interested in, it will suffice to show that $F(X)$ has P. Since, in this section we work mainly with $F(X)$, we will alter our hyperspace notation so that for a family of subsets $\mathcal{A}$ of X, $\langle\mathcal{A}\rangle$ will now represent $\{F \in F(X)\colon\ F \subset \bigcup\mathcal{A}$ and $F \cap A \neq \emptyset$ for every $A \in \mathcal{A}\}$.

A topological property P is said to be *summable* if it is preserved by topological sums, and it is *open-dense-extendible* if whenever X has P and X is open and dense in Y, then Y has P. A cardinal valued function α defined on the open subsets of a space X is *monotone* if $U \subset V$ implies that $\alpha(U) \leqslant \alpha(V)$. An open $U \subset X$ is *homogeneous by α* if for every nonempty open $V \subset U$, $\alpha(V) = \alpha(U)$.

Lemma 3.2. *Let α be a monotone cardinal valued function defined on the open subsets of a T_1 space X. Let P be a topological property that is finitely productive, summable and open-dense-extendible. If $F(U)$ has P for every open $U \subset X$ which is homogeneous by α, then $F(X)$ has P.*

Proof. Since α is monotone and there are no infinite decreasing sequences of cardinals, $\mathcal{P} = \{U\colon U$ is open in X and U is homogeneous by $\alpha\}$ is a π-base of X. Let $\mathcal{M}$ be a maximal disjoint subcollection of $\mathcal{P}$ ($\therefore \bigcup \mathcal{M}$ is dense in X). Put $\mathcal{U} = \{\langle \mathcal{F} \rangle\colon \mathcal{F}$ is a finite subcollection of $\mathcal{M}\}$. Then $\mathcal{U}$ is a disjoint open family in $F(X)$ such that $\bigcup \mathcal{U}$ is dense in $F(X)$. Since P is open-dense-extendible it suffices to show that $\bigcup \mathcal{U}$ has P. Since P is summable it suffices to show that for each finite $\mathcal{F} \subset \mathcal{M}$, $\langle \mathcal{F} \rangle$ has P. Let $\mathcal{F} = \{U_1, \ldots, U_n\}$. Then $\langle \mathcal{F} \rangle$ is homeomorphic to $\prod_{1 \leqslant i \leqslant n} F(U_i)$. Since P is finitely productive we have proved our lemma. $\square$

Theorem 3.3. *If X is a Hausdorff space which has a tree π-base, then $F(X)$ has a σ-disjoint π-base.*

Proof. Assume that X has a tree π-base. For each open $U \subset X$, put $h(U) = \min\{\mathrm{ht}(T)\colon T$ is a tree π-base of $U\}$. Then, h is a monotone cardinal valued function. Let P be the topological property of having a σ-disjoint π-base. P is ω-productive, summable and open-dense-extendible. By Lemma 3.2, it suffices to assume that X is homogeneous by h. Let T be a tree π-base of X of minimum height. Since X is homogeneous by h, we have the following basic property of T:

$$\forall t \in T\ \forall \gamma \text{ with } o(t) \leqslant \gamma < \mathrm{ht}(T)\ \exists s \in T \text{ with } s \subset t \text{ and } o(s) = \gamma. \qquad (*)$$

Since X is Hausdorff, if $\mathrm{ht}(T)$ is a successor ordinal, then $\mathrm{ht}(T) = 1$, X is discrete, and $F(X)$ is discrete; so we are done. Thus we assume that $\mathrm{ht}(T)$ is an infinite cardinal. For each $t \in T$ choose distinct $t_0, t_1 \subset t$ such that $o(t_0) = o(t) + 1 = o(t_1)$. Then t_0 and t_1 are disjoint and both are contained in t. For each $n \geqslant 1$, put $D_n = \{F \subset T\colon |F| = n$ and $\exists \alpha$ with $F \subset T_\alpha\}$. For each $F \in D_n$, put $\widehat{F} = \{t_0, t_1\colon t \in F\}$. Then, $\langle \widehat{F} \rangle$ is an open subset of $F(X)$. For each $n \geqslant 1$, put $\mathcal{D}_n = \{\langle \widehat{F} \rangle\colon F \in D_n\}$. We claim that $\mathcal{D} = \bigcup_{n<\omega} \mathcal{D}_n$ is a σ-disjoint π-base of $F(X)$. That $\mathcal{D}$ is a π-base follows from T being a tree π-base of X which has property $(*)$. Fix $n \geqslant 1$. Take $F \neq G$ in D_n. Let $F \subset T_\alpha$ and $G \subset T_\beta$. Without loss of generality, assume that $\alpha \leqslant \beta$. If there exists $t \in F$ such that $t \cap \bigcup G = \emptyset$, then $\langle \widehat{F} \rangle \subset [t]$ and $\langle \widehat{G} \rangle \cap [t] = \emptyset$ so $\langle \widehat{F} \rangle$ and $\langle \widehat{G} \rangle$ are disjoint. Otherwise, for every $t \in F$ there exists $s(t) \in G$ with $s(t) \subset t$. This map s is 1–1 on F since F is an antichain in the tree T. So, s is also onto G since $|F| = |G|$. Fix a $t \in F$. It now follows that either $t_0 \cap \bigcup G = \emptyset$ or $t_1 \cap \bigcup G = \emptyset$. Hence, either $[t_0] \cap \langle \widehat{G} \rangle = \emptyset$ or $[t_1] \cap \langle \widehat{G} \rangle = \emptyset$. $\langle \widehat{F} \rangle \subset [t_0] \cap [t_1]$, so $\langle \widehat{F} \rangle$ and $\langle \widehat{G} \rangle$ are disjoint. $\square$

Corollary 3.4. *If X is a T_4 space with a tree π-base, then $H(X)$ has a σ-disjoint π-base.*

Proof. By Theorem 3.3, $F(X)$ has a σ-disjoint π-base $\mathcal{P}$. Since X is T_4, $H(X)$ is T_3 and so the map $P \mapsto \mathrm{int}(\overline{P})$ sends $\mathcal{P}$ onto a σ-disjoint π-base of $H(X)$. $\square$

Proof. By Theorem 3.3, $F(X)$ has a σ-disjoint π-base $\mathcal{P}$. Since X is T_4, $H(X)$ is T_3 and so the map $P \mapsto \operatorname{int}(\overline{P})$ sends $\mathcal{P}$ onto a σ-disjoint π-base of $H(X)$. □

A fundamental space in topology is ω^*, the Stone–Čech remainder of the countable discrete space ω. A marvellous fact, due independently to Balcar, Pelant, and Simon [5] and Williams [11], is that ω^* has a tree π-base. Of course, ω^* is not d-separable (being a compact Hausdorff almost P space with no isolated points). Our result shows that $H(\omega^*)$ not only is d-separable but has a σ-disjoint π-base.

Question 3.5. Let X be a compact Hausdorff space. Does $H(X)$ have a σ-disjoint π-base if and only if X has a tree π-base?

Theorem 3.6. *If X is a locally compact Hausdorff space, then $F(X)$ is d-separable.*

Proof. Assume that X is locally compact and Hausdorff. For each open $U \subset X$, put $w(U) = \min\{|\mathcal{B}|\colon \mathcal{B} \text{ is a base of } U\}$. Then, w is a monotone cardinal valued function. Let P be the topological property of being d-separable. P is productive, summable and dense-extendible. By Lemma 3.2, it suffices to assume that X is homogeneous by w. If X is a 1-point space, then so is $F(X)$ and we are done. Otherwise, let $\mathrm{w}(X) = \kappa \geqslant \omega$ and let $\mathcal{B}$ be an open base with $|\mathcal{B}| = \kappa$ such that $\mathrm{w}(B) = \kappa$ for every $B \in \mathcal{B}$. For each positive integer n, let $\mathcal{C}_n = \{\langle\mathcal{U}\rangle\colon \mathcal{U} \text{ is a disjoint subcollection of } \mathcal{B} \text{ of cardinality } n\}$. Clearly, $\mathcal{C} = \bigcup_n \mathcal{C}_n$ is a base for $F(X)$. Our goal will be to show that $\forall n\ \exists$ a discrete $D_n \subset F(X)$ such that $\forall C \in \mathcal{C}_n\ \exists d \in D_n$ with $d \in C$.

Fix a positive integer n. Enumerate $\mathcal{C}_n$ as $\{\langle\mathcal{U}_\alpha\rangle\colon \alpha < \kappa\}$. Assume that for some of the β's with $\beta < \alpha < \kappa$ (we will call these β's relevant β's) we have defined sets $F_\beta \subset X$ of cardinality $n+1$ and $\mathcal{G}_\beta \in \mathcal{C}_{n+1}$ such that for $\beta \neq \gamma$ (both relevant) we have that

(1) $F_\beta \in \langle\mathcal{G}_\beta\rangle \cap \langle\mathcal{U}_\beta\rangle$ and

(2) $F_\beta \notin \langle\mathcal{G}_\gamma\rangle$.

At stage α, if there exists $\beta < \alpha$ such that $F_\beta \in \langle\mathcal{U}_\alpha\rangle$, then α is *not* relevant and proceed to stage $\alpha + 1$. Otherwise, we have that for every $\beta < \alpha$, $F_\beta \notin \langle\mathcal{U}_\alpha\rangle$. Choose $U \in \mathcal{U}_\alpha$ and a nonempty open V with $\overline{V} \subset U$ and $\overline{V}$ compact. Since $\mathrm{w}(\overline{V}) = \kappa$, Fact 1.1(a) implies that $\{W \cap \overline{V}\colon \beta < \alpha,\ \beta \text{ relevant, and } W \in \mathcal{G}_\beta\}$ does not T_1-separate the points of $\overline{V}$. Choose $x \neq y$ in $\overline{V}$ which witnesses this fact. Choose disjoint open subsets U_x and U_y of U with $x \in U_x$, $y \in U_y$ and $\{U_x, U_y\} \subset \mathcal{B}$. For each $R \in \mathcal{U}_\alpha$ pick $z(R) \in R$. Put $F_\alpha = \{x, y\} \cup \{z(R)\colon R \in \mathcal{U}_\alpha \text{ and } R \neq U\}$. Put $\mathcal{G}_\alpha = \{U_x, U_y\} \cup \{R \in \mathcal{U}_\alpha\colon R \neq U\}$. Thus we have inductive assumption (1) for α. To check (2) let $\beta < \alpha$ with β relevant. $F_\beta \notin \langle\mathcal{G}_\alpha\rangle$ because $F_\beta \notin \langle\mathcal{U}_\alpha\rangle$ and $\langle\mathcal{G}_\alpha\rangle \subset \langle\mathcal{U}_\alpha\rangle$. $F_\alpha \notin \langle\mathcal{G}_\beta\rangle$ because $|\mathcal{G}_\beta| = n + 1 = |F_\alpha|$, $\mathcal{G}_\beta$ is a disjoint collection and for every $W \in \mathcal{G}_\beta$ either $\{x, y\} \subset W$ or $\{x, y\} \cap W = \emptyset$. Finally, putting $D_n = \{F_\alpha\colon \alpha < \kappa \text{ and } \alpha \text{ relevant}\}$ we have achieved our goal. □

Corollary 3.7. *For locally compact Hausdorff spaces X, $H(X)$ is d-separable.*

A *π-net* $\mathcal{N}$ for a space X is a collection of nonempty subsets of X such that for every nonempty open set U, there exists $N \in \mathcal{N}$ with $N \subset U$.

Corollary 3.8. *Every locally compact Hausdorff space X has a π-net $\mathcal{N}$ consisting of sets of cardinality at most 2 such that $\mathcal{N}$ is a discrete subspace of $H(X)$.*

Proof. Choose a maximal disjoint subcollection $\mathcal{M}$ of a π-base consisting of sets which are homogeneous by w. It suffices to prove our corollary for each $M \in \mathcal{M}$ because, taking the union, over all $M \in \mathcal{M}$, of the sets produced gives us the desired π-net for X. If M is a singleton isolated point, then we are done; otherwise we just note that in Theorem 3.6, to meet open sets of the simple form $\langle U \rangle$, where U is a nonempty open subset of M, we used 2-element sets in our discrete subspace of $F(M)$. These 2-element sets form a π-net for M. $\square$

Most Tychonoff spaces X satisfy the conclusion of Corollary 3.8; indeed, we do not know of an honest counterexample; but at least we have the following consistent counterexample, courtesy of Stephen Watson: Let X be a nonseparable Tychonoff space such that X^2 has countable spread (such spaces are known to consistently exist). The map f from X^2 to $R = \{F \in H(X)\colon |F| \leqslant 2\}$ defined by $f((x,y)) = \{x,y\}$ is a continuous surjection. A discrete subspace D of R must be countable but then $\bigcup D$ is not dense in X. So X cannot have a π-net as in Corollary 3.8.

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